Math 436 (Spring 2020) - Homework 4

1. Which of the following sets are compact?
   (a) The set $\mathbb{Z}$ of integers in $\mathbb{R}$.
   (b) $\{1/n : n \in N_+\}$
   (c) The set $\{(x, y) \in \mathbb{R}^2 \mid y = \cos x, x \in [0, 1]\}$ in $\mathbb{R}^2$.
   (d) The set $\{(x, y) \in \mathbb{R}^2 \mid y = \tan x, x \in [0, \pi/2)\}$ in $\mathbb{R}^2$.

2. Show that $(0, 1)$ is not compact.

   Hint: compare $(0, 1)$ with the real line $\mathbb{R}$.

3. Show that $(0, 1)$ is not homeomorphic to $[0, 1]$.

4. Show that the subset $\mathbb{Q} \cap [0, 1]$ of $[0, 1]$ is not compact.

5. **Chapter 3:** 9

   Hint: adapt the proof of Theorem 3.6

6. Let A be a compact subset of a metric space X. Given $x \in X$, show that $d(x, A) = d(x, y)$ for some $y \in A$. Given a closed subset B, disjoint from A, show that $d(A, B) > 0$.

   Here $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$.

   Hint: Consider the function $f : X \to \mathbb{R}$ defined by $f(y) = d(x, y)$ and $g : X \to \mathbb{R}$ defined by $g(y) = d(y, B)$. Lemma 2.13 should be useful.